A_1-REGULARITY AND BOUNDEDNESS OF CALDERON-ZYGMUND OPERATORS

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ABSTRACT. The Coifman-Fefferman inequality implies quite easily that a Calderon-Zygmund operator T acts boundedly in a Banach lattice X on $\mathbb{R}^n$ if the Hardy-Littlewood maximal operator M is bounded in both X and X'. In this paper we discuss this phenomenon in some detail and establish a converse result under the assumption that X is p-convex and q-concave with some $1 < p, q < \infty$ and satisfies the Fatou property: if a linear operator T is bounded in X and T is nondegenerate in a certain sense (for example, if T is a Riesz projection) then M has to be bounded in both X and X'.

0. INTRODUCTION

The problem of characterizing spaces in which (and between which) the operators of harmonic analysis act boundedly is, perhaps, the central one in the modern harmonic analysis, and it definitely has far-reaching consequences in terms of applications. These operators in a vast number of cases can be represented by (or the corresponding questions reduced to the study of) a general Calderon-Zygmund operator. The study of such operators has received a lot of attention over the past several decades and significant advancements have been made. To mention a few highlights: the quest for practical conditions that guarantee boundedness of a Calderon-Zygmund operator in L_2 led to useful T_1 theorems, new approaches to the classical proofs allowed to significantly relax the doubling condition on the underlying measurable space, the action of such operators was studied in detail in a wide variety of spaces beyond the classical Lebesgue spaces L_p, and a number of representations for such operators was developed together with highly refined techniques that recently allowed to solve several long-standing open problems such as the A_2-hypothesis (positively) and the A_1 conjecture of Muckenhoupt and Wheeden (negatively). Although it seems that the focus has always been on particular classes of spaces, weighted Lebesgue spaces $L_p(w)$ being of a particular interest (not least because of their rather general nature which has long been noted), results extending various useful relationships to fairly general classes of spaces,

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and indeed sometimes demonstrating exhaustively the true scope of what has been known for many years, recently began to emerge.

In this paper we establish the following result that links together boundedness of Calderon-Zygmund singular integral operators T and boundedness of the Hardy-Littlewood maximal function M.

Theorem 1. *Suppose that X is a Banach lattice of measurable functions on $\mathbb{R}^n$ that satisfies the Fatou property and X is p-convex and q-concave with some $1 < p, q < \infty$. The following conditions are equivalent.*

(1) *The Hardy-Littlewood maximal operator M acts boundedly in X and in the order dual X' of X.*
(2) *All Calderon-Zygmund operators (with the standard smoothness conditions on the kernel) act boundedly in X.*
(3) *Any single Riesz projection R_j acts boundedly in X.*

We will explore several proofs of implication $1 \Rightarrow 2$ in Section 3 below. Implication $2 \Rightarrow 3$ is trivial, and implication $3 \Rightarrow 1$ will be established in Section 6. As it will be seen, sufficiency of Condition 3 of Theorem 1 for the other conditions actually extends to a wide class of singular operators that are nondegenerate in a certain sense. The proof, which is covered by Proposition 8 in Section 3 and in Theorem 23 in Section 6 below can easily be generalized to the case of a general space of homogeneous type instead of just $\mathbb{R}^n$ if there exists a suitable nondegenerate operator that can play the part of the Riesz projection R_j. It is easy to see that the proof of Theorem 1 also works in the vector-valued case, i. e. for lattices of measurable functions like $X(l^r)$, where X is a lattice on $\mathbb{R}^n$. The p-convexity and q-concavity assumptions are probably not necessary (they are not used in the implication $1 \Rightarrow 2$) and I conjecture that they in themselves are a consequence of any of the conditions of Theorem 1; that Condition 1 implies p-convexity and q-concavity with some $1 < p, q < \infty$ is known to hold true at least in the case of the variable exponent Lebesgue spaces (see, e. g., [5, Theorem 4.7.1]). It is also interesting to note that implication $1 \Rightarrow 2$ easily extends in a certain way to the case of operators acting between different Banach lattices; see Theorem 11 in Section 3 below.

Let us briefly outline some the contributions that led to this result. In the classical results describing properties of the Calderon-Zygmund operators (see, e. g., [27]) in the Lebesgue space setting the maximal operator plays an essential part. In the case of weighted Lebesgue spaces $\mathrm{L}_p(w)$ Theorem 1, of course, follows from the theory of the Muckenhoupt weights (see, e. g., [27, Chapter 5]) that individually links the conditions of Theorem 1 to the Muckenhoupt condition of

the weight w. Of a particular interest in this regard is the Coifman-Fefferman inequality [3]

$$\int |Tf|^p \omega \leqslant C \int (Mf)^p \omega, \quad 0 < p < \infty, \tag{1}$$

with C independent of f, that holds true for Calderon-Zygmund operators T (with the standard conditions on the kernel) and any weight $\omega \in \mathrm{A}_\infty$ for all locally summable functions f such that the right-hand part of (1) is finite. Thus T is estimated in terms of M for a relatively wide class of Muckenhoupt weights even though M may not act boundedly in the corresponding weighted Lebesgue space. There is a large number of various extensions of (1); see, e. g., [4]. On the other hand, making use of the duality and the famous construction due to Rubio de Francia allowed a large number of very useful extrapolation results that essentially exploit a very simple idea: if M is bounded in X then any $f \in X$ can be pointwise dominated with a controlled increase of norm by some weight $w \in \mathrm{A}_1$, and the converse is also trivially true. It is natural to call such lattices A_1-*regular* (see [25] for the history of this term). For example, this idea works very well in the case of variable exponent Lebesgue spaces $\mathrm{L}_{p(\cdot)}$ where the behavior of boundedness of M under duality and certain scaling operations is nice and well understood; see, e. g., [5, §7.2]. The Coifman-Fefferman inequality (1) with $p = 1$ gives a very easy proof of the implication $1 \Rightarrow 2$ of Theorem 1; see Proposition 8 in Section 3 below. And this is far from the only way to establish this implication; we will also discuss in Section 3 below how some of the recent results of A. Lerner [17], [18] and [19] also give the necessary tools to effortlessly establish this implication.

The study of the duality of BMO-regularity, initially motivated by certain problems in the theory of interpolation of Hardy-type spaces, eventually led in [25] to a refinement and generalization to the general spaces of homogeneous type of certain properties and results concerning the interplay of various majorization and boundedness properties that were previously known only in the case of the unit circle $\mathbb{T}$. In particular, the main result of [25] is similar to Theorem 1 because it links boundedness of T and M in lattices of the form $X^\alpha \mathrm{L}_1^{1-\alpha}$ for $0 < \beta < 1$ and sufficiently small $0 < \alpha < 1$ to another property, namely to BMO-regularity of X. This, of course, still left much more to be desired in terms of refinements because unlike A_1-regularity the BMO-regularity property, which proved to be very useful in certain questions pertaining to spaces on the unit circle $\mathbb{T}$, does not seem to be as useful in the case of the spaces on $\mathbb{R}^n$ in the same capacity. In this paper we will see how the techniques described in [25] can be adapted to establish the converse implications of Theorem 1.

The paper is organized as follows. In Section 1 we introduce some basic notions pertaining to Banach lattices and spaces of homogeneous

type. In Section 2 some known facts about Muckenhoupt weights and A_p-regular spaces are outlined. In Section 3 we briefly describe Calderon-Zygmund operators and show several ways to obtain the implication $1 \Rightarrow 2$ of Theorem 1. Then in Section 4 we discuss some results having to do with operators that are nondegenerate in a certain sense. Section 5 contains a new result that gives a sufficient condition for a lattice X to be A_1-regular in terms of A_1-regularity of lattice X^δ and A_p-regularity of lattice X. Finally, in Section 6 we prove the converse implication $3 \Rightarrow 1$ of Theorem 1.

1. Preliminaries

In this section we introduce some basic notions. Suppose that (S, ν) is a space of homogeneous type, i. e. S is a quasimetric space equipped with a Borel measure ν that has the doubling property:

$$\nu(B(x, 2r)) \leqslant c\nu(B(x, r))$$

for all $x \in S$ and $0 < r < \infty$ where c is a constant called a doubling constant. Futher information on these spaces and real harmonic analysis on them see e. g. in [6, 27]. It is customary in the literature on real harmonic analysis to consider only the typical case of S being a Euclidean space $\mathbb{R}^n$ equipped with the Lebesgue measure. However, for the main results of this paper this restriction is not at all necessary. We are working with (real or complex) quasi-normed lattices of measurable functions on $(S \times \Omega, \nu \times \mu)$ where (Ω, μ) is an arbitrary measurable space with a σ-finite measure μ. The second variable $t \in \Omega$ allows us to treat vector-valued results in this setting and it does not usually cause much trouble in the arguments. The case of just one variable x is, of course, included in this generality: one just takes the point mass measure for μ. A quasi-normed lattice of measurable functions X is by definition a quasi-normed space of measurable functions X in which the norm is compatible with the natural order; that is, if $|f| \leqslant g$ a. e. for some function $g \in X$ then $f \in X$ and $\|f\|_X \leqslant C\|g\|_X$ for some constant C independendent of f and g. One usually has $C = 1$; for example, it is always the case when the lattice has the Fatou property which will be introduced shortly. Further information on lattices can be found in e. g. [11, Chapter 10]. For simplicity we only work with lattices X such that $\operatorname{supp} X = S \times \Omega$. Many interesting normed spaces appearing in analysis are or can be represented as such lattices; for example the Lebesgue spaces L_p, the Orlicz spaces L_M, and the Lebesgue spaces $\mathrm{L}_{p(\cdot)}$ with variable exponent $p(\cdot)$ along with general modular spaces. The second variable allows us to naturally include lattices with mixed norm such as $\mathrm{L}_p(\mathrm{L}_r)$ in this setting.

Now we introduce certain properties of quasi-normed lattices and some related objects. For a Banach lattice of measurable functions X, any order continuous functional f on X (order continuity means that

given a sequence $x_n \in X$ such that $\sup_n |x_n| \in X$ and $x_n \to 0$ a. e. one also has $f(x_n) \to 0$) has an integral representation $f(x) = \int x y_f$ for some measurable function y_f which can be identified with f. The set of all such functionals X' is a Banach lattice with the norm defined by $\|f\|_{X'} = \sup_{g \in X, \|g\|_X = 1} \int |fg|$. The lattice X' is called the order dual of the lattice X. The norm of a lattice X is said to be order continuous if for any nonincreasing sequence $x_n \in X$ converging to 0 a. e. one also has $\|x_n\|_X \to 0$. The norm of a Banach lattice X is order continuous if and only if $X^* = X'$, i. e. if an only if all norm continous functionals on X are order continous. A lattice X has the Fatou property if for any $f_n, f \in X$ such that $\|f_n\|_X \leqslant 1$ and the sequence f_n converges to f a. e. it is also true that $f \in X$ and $\|f\|_X \leqslant 1$. The Fatou property of a lattice X is equivalent to $(\nu \times \mu)$-closedness of the unit ball B_X of the lattice X (here and elsewhere $(\nu \times \mu)$-convergence denotes the convergence in measure in any measurable set E such that $(\nu \times \mu)(E) < \infty$). If the lattice X is Banach then the Fatou property is equivalent to order reflexivity of X, i. e. to the relation $X'' = X$. For a lattice X either one of the Fatou property or the order continuity of norm property is sufficient to guarantee that the lattice X' is a norming set of functionals for X, i. e. that $\|f\|_X = \sup_{g \in X', \|g\|_{X'} = 1} \int fg$ for all $f \in X$. The order dual X' of a Banach lattice X always has the Fatou property.

We now introduce a couple of very useful constructions that play a major part in the analysis of lattices and their majorization and other properties. For any two quasi-normed lattices X and Y on the same measurable space the set of poinwise products of their functions

$$XY = \{fg \mid f \in X, g \in Y\}$$

is a quasi-normed lattice with the norm defined by

$$\|h\|_{XY} = \inf_{h=fg} \|f\|_X \|g\|_Y.$$

If both lattices X and Y satisfy the Fatou property then the lattice XY also has the Fatou property. This lattice multiplication is associative: if X, Y and Z are lattices of measurable functions on the same measurable space then $(XY)Z = X(YZ)$ (here and elsewhere, if not stated otherwise, the equality of lattices is understood as the equality of the sets together with the equality of the quasi-norms). It is easy to see that if either of the lattices X and Y has order continuous quasi-norm then the norm of the lattice XY is also order continuous.

For any $\delta > 0$ and a quasi-normed lattice X the lattice X^δ consists of all measurable functions f having well-defined and finite quasi-norms $\|f\|_{X^\delta} = \||f|^{1/\delta}\|_X^\delta$. For example, $\mathrm{L}_p^\delta = \mathrm{L}_{\frac{p}{\delta}}$. For any product XY of quasi-lattices and $\delta > 0$ one has the natural relation $(XY)^\delta = X^\delta Y^\delta$. If a lattice X has the Fatou property then X^δ also has the Fatou property; if a lattice X has order continous quasi-norm then X^δ also has order continuous quasi-norm. For any $0 < \delta \leqslant 1$, if X is a Banach lattice then

X^δ is also a Banach lattice. If X and Y are Banach lattices then for any $0 < \delta < 1$ lattice $X^{1-\delta} Y^\delta$, sometimes called the *Calderon-Lozanovsky product* of X and Y, is also Banach; moreover, one has a very useful relation $(X^{1-\delta} Y^\delta)' = (X')^{1-\delta} (Y')^\delta$ (see [2], [22]). If $Z = X^{1-\delta} Y^\delta$ either has the Fatou property or has order continuous norm then Z is an exact interpolation space of exponent δ between X and Y (which coincides with either the complex interpolation space $(X, Y)^\theta$ or with $(X, Y)_\theta$ respectively), i. e. any linear operator T that acts boundedly in X and in Y also acts boundedly in Z and $\|T\|_Z \leqslant \|T\|_X^{1-\delta} \|T\|_Y^\delta$; see, e. g., [23], [2], [14] for more detail.

Let $1 \leqslant p, q < \infty$. A Banach lattice X is said to be *p-convex* with constant C if

$$\left\| \left(\sum_{j=1}^N |f_j|^p \right)^{\frac{1}{p}} \right\|_X \leqslant C \left(\sum_{j=1}^n \|f_j\|_X^p \right)^{\frac{1}{p}}$$

for any $\{f_j\}_{j=1}^N \subset X$; X is said to be *q-concave* with constant c if

$$\left(\sum_{j=1}^n \|f_j\|_X^q \right)^{\frac{1}{q}} \leqslant c \left\| \left(\sum_{j=1}^N |f_j|^q \right)^{\frac{1}{q}} \right\|_X$$

for any $\{f_j\}_{j=1}^N \subset X$. If X is p-convex then X' is p'-concave, and if X is q-concave then X' is q'-convex. It is well known (see, e. g., [21, Book II, Proposition 1.d.8]) that a Banach lattice that is p-convex and q-concave can be renormed so that its p-convexity and q-concavity constants are both 1. The assumption of p-convexity imposed on a lattice X enables us to raise the lattice in a power $1 < p < \infty$ without it becoming quasi-Banach since p-convexity of X is equivalent to 1-convexity of $Y = X^p$. This in turn implies that $X = Y^{\frac{1}{p}}$ and $X' = (Y')^{\frac{1}{p}} \mathrm{L}_1^{1-\frac{1}{p}}$ provided that X has the Fatou property, so in this case X' has order continuous norm and therefore $X = X'' = X'^*$. By the same argument if a lattice X has the Fatou property and X is q-concave for some $1 < q < \infty$ then X has order continuous norm and $X' = X^*$. Thus a lattice X which is both p-convex and q-concave with some $1 < p, q < \infty$ is reflexive, and also both X and X' have order continuous norm and enjoy many other nice properties.

For a quasi-normed lattice X and weights w such that $0 \leqslant w \leqslant \infty$ almost everywhere the weighted lattice $X(w)$ is defined by

$$X(w) = \left\{ g \mid \frac{g}{w} \in X \right\}$$

with the quasi-seminorm defined by $\|f\|_{X(w)} = \|f w^{-1}\|_X$. This somewhat cumbersome definition is needed because the more natural definition

$$X(w) = \{ wh \mid h \in X \}$$

is meaningless if the weight w takes value $+\infty$ on a set of positive measure and it seems to be easier to allow this in the definition and work with weighted lattices that may be quasi-normed rather than negotiate finiteness of w every time. Thus in this setting one has $g = 0$ on the set where $w = 0$, g restricted on the set $\{w = +\infty\}$ is an arbitrary measurable function, and $\|\cdot\|_{X(w)}$ is a norm for weights w such that $\nu \times \mu(\{w = +\infty\}) = 0$. If $w = 0$ on a set of positive measure, we regard $X(w)$ as merely a set of functions with a seminorm under our conventions, since then $\operatorname{supp} X(w) \neq \operatorname{supp} X$. Note that in majorization arguments it is usually possible to avoid dealing with “bad” weights with the help of the following proposition.

Proposition 2 ([25, Proposition 3.2]). *Suppose that X is a Banach lattice on (Σ, μ). Then for every $f \in X$ such that $f \neq 0$ identically and $\varepsilon > 0$ there exists $g \in X$ such that $g > |f|$ a. e. and $\|g\|_X \leqslant (1+\varepsilon)\|f\|_X$.*

The construction of a weighted lattice yields

$$\mathrm{L}_\infty(w) = \{f \mid |f| \leqslant Cw \text{ a. e. }\}.$$

It is easy to see that $[X(w)]' = X'(w^{-1})$. Notice that this definition of the weighted Lebesgue space $\mathrm{L}_p(w)$ differs from the “classical” one with the norm defined by $\|f\|_{p,w}^p = \int |f|^p w$, which is often used in the literature; the latter norm corresponds to the norm of the lattice $\mathrm{L}_p\left(w^{-\frac{1}{p}}\right)$ in our notation. Thus all weighted lattices are defined in the same way everywhere in this paper; however, one has to pay attention to this difference. We adopt the natural conventions $0^{-1} = \infty$ and $\infty^{-1} = 0$ in all expressions involving weights.

2. Muckenhoupt weights and A_p-majorants

In this section we introduce some useful notions having to do with the Muckenhoupt weights; for more detail see, e. g., [27, Chapter 5]. The (centered) Hardy-Littlewood maximal operator

$$Mf(x,t) = \sup_{r>0} \frac{1}{\nu(B(x,r))} \int_{B(x,r)} |f(z,t)| d\nu(z), \quad x \in S, \quad t \in \Omega,$$

is well-defined for a. e. $x \in S, t \in \Omega$, and the measurable functions f on $(S \times \Omega, \nu \times \mu)$ that are locally summable in the first variable. We say that a non-negative measurable function w on $(S \times \Omega, \nu \times \mu)$ belongs to the Muckenhoupt class A_p for some $1 \leqslant p < \infty$ with a constant C if

$$\operatorname*{ess\,sup}_{t \in \Omega} \|M\|_{\mathrm{L}_p\left(w^{-1/p}(\cdot,t)\right) \to \mathrm{L}_{p,\infty}\left(w^{-1/p}(\cdot,t)\right)} \leqslant C.$$

In the case $p > 1$ this condition is equivalent to

$$\operatorname*{ess\,sup}_{t \in \Omega} \|M\|_{\mathrm{L}_p\left(w^{-1/p}(\cdot,t)\right)} \leqslant C'$$

with a constant C' estimated in terms of C and p. The class A_1 is characterized by the estimate $Mw \leqslant C'w$ almost everywhere, while classes A_p for $p > 1$ are characterized by the well-known Muckenhoupt condition

$$\operatorname*{ess\,sup}_{x\in S, t\in\Omega} \sup_{r>0} \left[\frac{1}{\nu\left(B(x,r)\right)} \int\limits_{B(x,r)} w(u,t) d\nu(u)\right] \times \left[\frac{1}{\nu\left(B(x,r)\right)} \int\limits_{B(x,r)} w(u,t)^{-\frac{1}{p-1}} d\nu(u)\right]^{p-1} < \infty. \tag{2}$$

The following notion is a natural refinement of the BMO-regularity property which was apparently first introduced by N. Kalton in [10].

Definition 3. *A quasi-normed lattice X on $(S\times\Omega, \nu\times\mu)$ is A_p-regular with constants (C, m) if for any $f \in X$ there exists a majorant $g \in X$, $g \geqslant |f|$ such that $\|g\|_X \leqslant m\|f\|_X$ and $g \in \mathrm{A}_p$ with constant C.*

This property was introduced and studied to some extent in [25]; we will reference here only the results used in the present work.

Proposition 4 ([25, Proposition 1.2])**.** *A quasi-normed lattice X on $(S \times \Omega, \nu \times \mu)$ is A_1-regular if and only if the maximal operator M is bounded in X.*

Sufficiency is trivial, and necessity quickly follows from application of the famous Rubio de Francia construction. The following proposition is a direct application of duality and the connection between Muckenhoupt weights and boundedness of the maximal operator.

Proposition 5 ([25, Proposition 2.3])**.** *Suppose that X is a Banach lattice on $(S \times \Omega, \nu \times \mu)$ such that X' is a norming space for X. If X' is A_1-regular then $X^{\frac{1}{q}}$ is A_1-regular for all $q > 1$. If X' is A_p-regular with some $p > 1$ then $X^{\frac{1}{p}}$ is A_1-regular.*

The following result was obtained in [25] in order to establish the division and self-duality property of BMO-regularity in the general setting.

Theorem 6 ([25, Theorem 1.6])**.** *Suppose that X is a Banach lattice on $(S\times\Omega, \nu\times\mu)$ having the Fatou property. Suppose also that $X\mathrm{L}_q$ for some $1 < q < \infty$ is a Banach lattice and $X\mathrm{L}_q$ is A_p-regular for some $1 \leqslant p < \infty$. Then X is A_{p+1}-regular.*

The only currently known method of establishing this result is through application of a fixed point theorem somewhat similar to what is done in Section 5 below.

As a consequence of the reverse Hölder inequality we see that the A_1-regularity property is self-improving, which is the subject of the

following proposition. (It is not difficult to see that the general A_p-regularity property is also self-improving in this manner, but we will not need it in the present work). There is also a fairly general approach that makes it possible to establish this property using certain methods stemming from the geometry of Banach spaces; see [20].

Proposition 7. *Suppose that X is an A_1-regular Banach lattice on $(S \times \Omega, \nu \times \mu)$ with constants (C, m). Then X^r is also an A_1-regular lattice for some $r > 1$ depending only on C.*

Indeed, let $r > 1$ be the constant of the reverse Hölder inequality that is satisfied for all A_1 weights with constant C. Suppose that $f \in X^r$, and let g be an A_1-majorant for $|f|^{\frac{1}{r}}$ in X with constants (C, m). Then g^r is an A_1-majorant for f with constants independent of f, because by the reverse Hölder inequality we have an estimate

$$\frac{1}{\nu(B(x,\rho))} \int_{B(x,\rho)} g^r(u,\omega) d\nu(u) \leqslant \\ c \left(\frac{1}{\nu(B(x,\rho))} \int_{B(x,\rho)} g(u,\omega) d\nu(u) \right)^r \leqslant c\, C^r [g(x,\omega)]^r$$

for almost all $x \in S$, $\omega \in \Omega$ and $\rho > 0$ with a constant c independent of f, x, ω and ρ.

3. Calderon-Zygmund operators

In this section we will show how certain conditions on the lattices are sufficient for boundedness of the Calderon-Zygmund operators in the general setting. Namely, we give 4 somewhat independent proofs of the implication $1 \Rightarrow 2$ of Theorem 1. For a standard survey on the singular integral operators see, e. g., [27]. Although it seems possible to extend all of the results used here to the general setting of spaces of homogeneous type and beyond, for simplicity we will only discuss the standard setting of $\mathbb{R}^n$ with the usual Lebesgue measure $d\nu = dm$. We will also use, in contrast to the definition introduced in Section 2, the (uncentered) Hardy-Littlewood maximal operator defined by

$$Mf(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)| dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ with sides parallel to the coordinate axes. However, it is well known that this definition is pointwise equivalent to the one given before.

We say that T is a *Calderon-Zygmund operator* if T is a singular integral operator that is bounded in $L_2(\mathbb{R}^n)$ and its kernel $K(x, y)$ satisfies

(3)

$$|K(x,s) - K(x,t)| \leqslant C_K \frac{|s-t|^\gamma}{|x-s|^{n+\gamma}}, \quad x, s, t \in \mathbb{R}^n, \quad |x-s| > 2|s-t|,$$

and the kernel $K(y, x)$ of the adjoint operator T^* satisfies the same estimates. It is well known that T is bounded in L_p for all $1 < p < \infty$. Although recently interesting new tools were developed for treatment of such operators in a fairly general setting, we set off with the more classical approach, which is also very simple. The following proposition contains the implication $1 \Rightarrow 2$ of Theorem 1.

Proposition 8. *Suppose that X is a Banach lattice on $\mathbb{R}^n$ having either the Fatou property or order continuous norm and both X and X' are A_1-regular. Then any Calderon-Zygmund operator T is bounded in X.*

Indeed, let $f \in X$ and $g \in X'$, and let h be an A_1-majorant of g in X'. Then

$$\int (Mf)h \leqslant \|Mf\|_X \|h\|_{X'} \leqslant c_1 \|f\|_X \|g\|_{X'} < \infty,$$

and the Coifman-Fefferman inequality (1) with $p = 1$ implies that

$$\int (Tf)g \leqslant \int |Tf| h \leqslant c \int (Mf)h \leqslant c\, c_1 \|f\|_X \|g\|_{X'}$$

with certain constants c and c_1 independent of f and g, which implies that T acts boundedly in X. Compared to the other approaches that follow (that, at least in their presently available form, significantly rely in their details on the structure of the dyadic cubes in $\mathbb{R}^n$ which makes it harder to carry the arguments over to a general space of homogeneous type), it is easy to see that the Coifman-Fefferman inequality and other parts of the proof remain valid in the general case of Calderon-Zygmund operators on σ-finite spaces of measurable functions on $S \times \Omega$ where S is a space of homogeneous type.

Now let us briefly describe another approach to establishing a slightly weaker version of Proopsition 8 that uses the classical Fefferman-Stein inequality that was recently generalized to general Banach lattices. The Fefferman-Stein maximal function $f^\sharp$ on $\mathbb{R}^n$ is defined for a locally integrable function f by

$$f^\sharp(x) = \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x with sides parallel to the coordinate axes and $f_Q = \frac{1}{|Q|} \int_Q f(z) dz$ is the average of f over Q with respect to the Lebesgue measure. This maximal function is very useful in the estimates of the Calderon-Zygmund operators T. On the one hand, we have well-known (and rather simple) pointwise inequality

$$(Tf)^\sharp \leqslant c_r \left(M|f|^r\right)^{\frac{1}{r}} \tag{4}$$

almost everywhere for any $1 < r < \infty$; see, e. g., [27, Chapter 4, §4.2]. On the other hand, there is the classical Fefferman-Stein inequality

$$\|f\|_{\mathrm{L}_p} \leqslant c\|f^\sharp\|_{\mathrm{L}_p}$$

for $1 < p < \infty$. The latter was recently generalized to general Banach lattices in the following manner (see [18] for more information). In the rather convenient notation S_0 denotes the set of all measurable functions f on $\mathbb{R}^n$ such that their nonincreasing rearrangement f^* satisfies $f^*(+\infty) = 0$, and the main tools of [18] that will appear shortly in this seciton work for this class of functions rather than just the locally summable ones. Surely S_0 contains all measurable functions with compact support. It is easy to see that S_0 is dense in a Banach lattice X if, for example, X has order continuous norm. The converse, however, is not true: take, for example, $X = \mathrm{L}_\infty(w)$ with a weight w satisfying $w^*(+\infty) = 0$.

Theorem 9 ([18, Corollary 4.3]). *Suppose that X is an A_1-regular real Banach lattice of measurable functions on $\mathbb{R}^n$ having the Fatou property. Then the following conditions are equivalent.*

(1) *X' is A_1-regular.*
(2) *There exists some $c > 0$ such that $\|f\|_X \leqslant c\|f^\sharp\|_X$ for all $f \in S_0 \cap X$.*

These two ingredients allow us to easily establish Proposition 8 under the additional assumption that S_0 is dense in X. Indeed, by the assumed density property it is sufficient to estimate $\|Tf\|_X$ for all $f \in S_0 \cap X$. An application of Theorem 9, (4) and Proposition 7 yields

$$\|Tf\|_X \leqslant c\left\|(Tf)^\sharp\right\|_X \leqslant c\,c_r\left\|(M|f|^r)^{\frac{1}{r}}\right\|_X \leqslant \\ c\,c_r\|M\|^{\frac{1}{r}}_{X^r\to X^r}\|f\|_X \leqslant c_1\|f\|_X$$

with some $r > 1$ and constants c, c_1 and c_r independent of f. The restriction that X is a real Banach lattice is easy to lift; see, e. g., [26, Proposition 6].

We need some more preliminaries before further discussion. The *Strömberg local sharp maximal function* is defined for $f \in S_0$ by

$$M^\sharp_\lambda f(x) = \sup_{Q\ni x}\inf_{c\in\mathbb{R}}((f-c)\chi_Q)^*(\lambda|Q|), \quad x \in \mathbb{R}^n, \tag{5}$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$ containing x. Functions $M^\sharp_\lambda f$ and $f^\sharp$ are closely related via the following estimate which holds true with all sufficiently small $0 < \lambda < 1$ and some $c_0, c_1 > 1$ for all locally summable f:

$$c_0 MM^\sharp_\lambda f(x) \leqslant f^\sharp(x) \leqslant c_1 MM^\sharp_\lambda f(x), \quad x \in \mathbb{R}^n; \tag{6}$$

see, e. g., [9], [16]. On the other hand, $M^\sharp_\lambda$ in many cases provides estimates that are significantly finer than those obtained via the Fefferman-Stein sharp maximal function. For example (see [1], [9]),

$$M^\sharp_\lambda(Tf) \leqslant cMf \tag{7}$$

almost everywhere for all locally summable functions f with c independent of f. Estimate (7) is sharper than (4), as it corresponds to the missing limiting case $r = 1$ in (4), and we can easily obtain (4) from (7) using (6) and [27, Chapter 5, §5.2]. Finally, there is the following result similar to the well-known duality relation between H_1 and BMO.

Theorem 10 ([17, Theorem 1]).

$$\int |fg| \leqslant c \int M^\sharp_\lambda f \, Mg$$

for any $f \in S_0$ and locally summable function g with some c and λ independent of f and g.

Examining the details of the previous argument contained in Theorem 9 quickly leads to the following observation.

Theorem 11. *Suppose that X, Y and Z are Banach lattices on $\mathbb{R}^n$ having the Fatou property, S_0 is dense in X, and suppose that the Hardy-Littlewood maximal operator M acts boundedly from X to Z and from Y' to Z'. Then any operator T that satisfies estimate (7) acts boundedly from X to Y.*

Theorem 11 follows at once from Theorem 10, since for any $f \in X \cap S_0$ and $g \in Y'$ we have an estimate

$$\int |Tf\, g| \leqslant c \int M^\sharp_\lambda(Tf)\, Mg \leqslant c_1 \int Mf\, Mg \leqslant \\ c_1 \|Mf\|_Z \|Mg\|_{Z'} \leqslant c_2 \|f\|_X \|g\|_{Y'} \tag{8}$$

with some c, c_1 and c_2 independent of f and g.

Since M is a positive operator and $Mg \geqslant g$ almost everywhere for any locally summable g, conditions of Theorem 11 imply that $X \subset Z$ and $Y' \subset Z'$ in the sense of continuous inclusions, which in turn implies that $X \subset Z \subset Y$. Unlike the case $X = Y = Z$ it is presently unclear whether Theorem 11 admits a converse similar to Theorem 23 below. In other words, if a suitable Calderon-Zygmund operator T acts boundedly from X to Y, does it follow that there exists a lattice Z satisfying the conditions of Theorem 11?

There are still other approaches to establishing estimate (8). Let us now describe a recent result [19]. First, we need some preliminaries. If T is a Calderon-Zygmund operator with kernel K then there is a maximal truncation operator

$$T_\natural f(x) = \sup_{0<\varepsilon<A} \left| \int_{\varepsilon<|y|<\nu} K(x,y) f(y)\, dy \right|, \quad x \in \mathbb{R}^n,$$

associated with T defined for all locally summable functions f. It is well known (see, e. g., [27, Chapter 1, §7]) that this operator is bounded in L_p for all $1 < p < \infty$. The maximal truncated operator $T_\natural$ dominates the family of truncations

$$T_{\varepsilon,A}f(x) = \int_{\varepsilon<|y|<\nu} K(x,y)f(y)\,dy, \quad x \in \mathbb{R}^n,$$

of T, so this family of operators has a weak limit T_* in L_2 as $\varepsilon \to 0$ and $A \to \infty$, and there exists some $a \in \mathrm{L}_\infty$ such that

$$Tf(x) = T_*f(x) + a(x)f(x)$$

for all $f \in \mathrm{L}_2$ and almost all $x \in \mathbb{R}^n$. Since multiplication by a bounded function a is bounded in any lattice, boundedness of T in a given lattice X is thus implied by boundedness of the maximal truncation operator $T_\natural$.

A *dyadic grid* $\mathcal{D}$ is a collection of cubes Q in $\mathbb{R}^n$ with sides parallel to the coordinate axes such that their lengths $\ell(Q)$ are of the form 2^k, $k \in \mathbb{Z}$, for any $Q, R \in \mathcal{D}$ we have $Q \cap R \in \{Q, R, \emptyset\}$, and the cubes $\{Q \in \mathcal{D} \mid \ell(Q) = 2^k\}$ form a partition of $\mathbb{R}^n$ for any $k \in \mathbb{Z}$. A collection $\mathcal{S} = \{Q_j^k\} \subset \mathcal{D}$ is called a *sparse family* of dyadic cubes if it satisfies the following properties.

(1) Cubes Q_j^k are pairwise disjoint in j with k fixed.
(2) If $\Omega_k = \bigcup_j Q_j^k$ then $\Omega_{k+1} \subset \Omega_k$.
(3) $|\Omega_{k+1} \cap Q_j^k| \leqslant \frac{1}{2}|Q_j^k|$ for any j and k.

For any family of cubes $\mathcal{S}$ we define an operator

$$\mathcal{A}_{\mathcal{D},\mathcal{S}}f(x) = \mathcal{A}_{\mathcal{S}}f(x) = \sum_{Q\in\mathcal{S}} f_Q \chi_Q(x)$$

acting on locally summable functions f. It turns out that these operators with sparse families can be used to estimate Calderon-Zygmund operators in the general setting.

Theorem 12 ([19, Theorem 1.1])**.** *Suppose that X is a Banach lattice on $\mathbb{R}^n$ having the Fatou property. Then*

$$\|T_\natural f\|_X \leqslant c_{T,n} \sup_{\mathcal{D},\mathcal{S}} \|\mathcal{A}_{\mathcal{D},\mathcal{S}}|f|\|_X$$

for any Calderon-Zygmund operator T and locally summable function f with compact support, where the supremum is taken over arbitrary dyadic grids $\mathcal{D}$ and sparse families $\mathcal{S} \subset \mathcal{D}$.

Theorem 12 is based on a number of results that only recently were developed to a sufficient extent, including a representation of Calderon-Zygmund operators as an average of dyadic shifts and the local mean oscillation decomposition that represents every function $f \in S_0$ as $\mathcal{A}_{\mathcal{S}}f$ for some sparse family $\mathcal{S}$ in a given dyadic grid $\mathcal{D}$ with good pointwise control on $f - \mathcal{A}_{\mathcal{S}}f$; see [19] for a brief history history of the techniques.

As it was shown in [19], Theorem 12 has many interesting corollaries, including the so-called A_2 conjecture and certain two-weight estimates. Let us verify a replacement for the first line of the estimate (8) for a suitable function $f \in X$ with operator $T_\natural$ in place of T. By Theorem 12 there exists a dyadic grid $\mathcal{D}$ and a sparse family $\mathcal{S} \subset \mathcal{D}$ such that $\|T_\natural f\|_Y \leqslant c\|\mathcal{A}_{\mathcal{D},\mathcal{S}}|f|\|_Y$ with some constant c independent of f. Therefore there exist some $g \in Y'$, $\|g_0\|_{Y'} \leqslant 1$, such that

$$\|T_\natural f\|_Y \leqslant 2c \int (A_{\mathcal{D},\mathcal{S}}|f|)\, g. \tag{9}$$

We may assume that $g \geqslant 0$. The integral on the right-hand side of (9) can now be easily estimated using a kind of a stopping time argument [19, (2.2)] which we are going to reproduce here. Let $\{Q_j^k\} = \mathcal{S}$ and Ω_k be the cubes and sets in the definition of the sparse family $\mathcal{S}$, and let $E_j^k = Q_j^k \setminus \Omega_{k+1}$, so that $|E_j^k| \geqslant \frac{1}{2}|Q_j^k|$ and $\{E_j^k\}$ is a collection of pairwise disjoint sets. Then

$$\begin{aligned}
\int (\mathcal{A}_{\mathcal{D},\mathcal{S}}|f|)\, g = \sum_{j,k} \frac{1}{|Q_j^k|} \int_{Q_j^k} |f| \int_{Q_j^k} g = \\
\sum_{j,k} |Q_j^k| \frac{1}{|Q_j^k|} \int_{Q_j^k} |f| \frac{1}{|Q_j^k|} \int_{Q_j^k} g \leqslant \\
2\sum_{j,k} |E_j^k| \frac{1}{|Q_j^k|} \int_{Q_j^k} |f| \frac{1}{|Q_j^k|} \int_{Q_j^k} g = \\
2\sum_{j,k} \int \chi_{E_j^k} \frac{1}{|Q_j^k|} \int_{Q_j^k} |f| \frac{1}{|Q_j^k|} \int_{Q_j^k} g \leqslant \\
2\sum_{j,k} \int_{E_j^k} Mf\, Mg \leqslant 2 \int Mf\, Mg,
\end{aligned}$$

which together with (9) is a suitable replacement for the first line in estimate (8). Another estimate for Calderon-Zygmund and certain other operators that can also be used to establish (8) can be found in [8].

4. Nondegenerate singular operators

In this section we will try to give a more or less precise meaning to the nondegeneracy conditions that a singular operator in Condition 2 of Theorem 1 must satisfy in order to have a converse implication $2 \Rightarrow 1$ as well as discuss certain restrictions on the spaces that are implied by boundedness of certain classes of operators. Recall that the Muckenhoupt weights $w \in \mathrm{A}_2$ are precisely those for which the Hardy-Littlewood maximal operator is bounded in the corresponding weighted space $\mathrm{L}_2\left(w^{-\frac{1}{2}}\right)$. There is, however, a large class of operators that also characterize Muckenhoupt weights in this sense.

Definition 13. *A mapping $T : L_2 \to L_2$ is called* A_2*-nondegenerate with a constant C if boundedness of T in a lattice $L_2\left(w^{-\frac{1}{2}}\right)$ implies $w \in A_2$ with constant C.*

This definition is stated for the general setting of a homogeneous space S and measurable functions on $S\times\Omega$. It is worth mentioning that for linear maps $Tf(x,y) = [T_0 f(\cdot,y)](x)$ that act in the first variable $x \in S$ only, i. e. uniformly in $y \in \Omega$, nondegeneracy of T_0 on S implies nondegeneracy of T on $S \times \Omega$; see [25, Proposition 3.7]. For simplicity we will only work with a single variable from $S = \mathbb{R}^n$ in this section.

Although it is not clear yet how nondegeneracy in the sense of Definition 13 can be characterized in terms of the kernel of a singular integral operator T, there are some useful sufficient conditions that illustrate this phenomenon.

Definition 14. *We say that a mapping $T : L_2 \to L_2$ is nondegenerate along a direction $x_0 \in \mathbb{R}^n \setminus \{0\}$ if there exists a constant $c > 0$ such that for any ball $B \subset \mathbb{R}^n$ of radius $r > 0$ and any locally summable nonnegative function f supported on B we have*

$$|Tf(x)| \geqslant cf_B \tag{10}$$

for all $x \in B \pm rx_0$.

It is well known that singularity of a mapping T in the sense of Definition 14 implies A_2-nondegeneracy of T; in Proposition 15 below we will establish a somewhat more general result. In terms of the kernel K of T condition (10) says roughly that $K(x,y)$ in terms of $x-y$ has to increase at 0 as quickly and decay at infinity as slowly as $|x-y|^{-n}$ along a certain direction; this statement is made more precise in Proposition 18 below. It is not clear whether the class of mappings described by Definition 13 is actually more general than that described by Definition 14.

Let $\mathcal{S} = \{Q_l\}$ be a collection of cubes or balls. In addition to $\mathcal{A}_\mathcal{S}$ we introduce the following averaging operator

$$\mathcal{A}_\mathcal{S}^\square f(x) = \left(\sum_{Q\in\mathcal{S}} (f_Q)^2 \chi_Q(x)\right)^{\frac{1}{2}}$$

for all locally summable functions f. It is easy to see that if the cubes or balls from $\mathcal{S}$ are pairwise disjoint then $\mathcal{A}_\mathcal{S}^\square f = \mathcal{A}_\mathcal{S} f$ almost everywhere for nonnegative functions f.

Proposition 15. *Suppose that a linear operator T that is nondegenerate along a direction x_0 is bounded with norm C in a Banach lattice X having the Fatou property. Then for any collection of cubes or balls*

$\mathcal{S} = \{Q_l\}$ we have

$$\|\mathcal{A}^{\square}_{\mathcal{S}} f\|_X \leqslant c_a \left\| f \left(\sum_l \chi_{Q_l} \right)^{\frac{1}{2}} \right\|_X \tag{11}$$

for all f such that the right-hand part of (11) *is well-defined with a constant c_a independent of f and $\mathcal{S}$.*

To prove Proposition 15 let $\mathcal{S}' = \{Q'_l\}$ with $Q'_l = Q_l + x_0$ being the cubes or balls Q_l shifted by x_0 and set $f_l = f\chi_{Q_l}$. We may assume that f is nonnegative and that the right-hand part of (11) is finite. It follows that the sequence valued function $F = \{f_l\}$ belongs to $X(l^2)$ with $\|F\|_{X(l^2)} = \left\| f \left(\sum_l \chi_{Q_l}\right)^{\frac{1}{2}} \right\|_X$. Using the nondegeneracy assumption and the Grothendieck theorem (see, e. g., [15]) we can easily obtain the estimate

$$c^{-1} \left\| \left(\sum_l \chi_{Q'_l} (f_{Q_l})^2 \right)^{\frac{1}{2}} \right\|_X \leqslant \left\| \left(\sum_l \chi_{Q'_l} |T f_l|^2 \right)^{\frac{1}{2}} \right\|_X \leqslant \\ \|TF\|_{X(l^2)} \leqslant C K_G \|F\|_{X(l^2)} = C K_G \left\| f \left(\sum_l \chi_{Q_l} \right)^{\frac{1}{2}} \right\|_X , \tag{12}$$

K_G being the Grothendieck constant. Repeating this estimate for function $G = \{g_l\}$, $g_l = \chi_{Q'_l} f_{Q_l}$, in place of F and with the order of Q_l and Q'_l reversed yields

$$c^{-1} \|\mathcal{A}^{\square}_{\mathcal{S}} f\|_X = c^{-1} \left\| \left(\sum_l \chi_{Q_l} (f_{Q_l})^2 \right)^{\frac{1}{2}} \right\|_X \leqslant \\ \left\| \left(\sum_l \chi_{Q_l} |T g_l|^2 \right)^{\frac{1}{2}} \right\|_X \leqslant \|TG\|_{X(l^2)} \leqslant C K_G \|G\|_{X(l^2)} = \\ C K_G \left\| \left(\sum_l \chi_{Q'_l} (f_{Q_l})^2 \right)^{\frac{1}{2}} \right\|_X . \tag{13}$$

Combining (12) and (13) together yields (11) with $c_a = (CcK_G)^2$.

The following corollary is essentially well known; see, e. g., remarks after [5, Lemma 5.2.2].

Corollary 16. *Suppose that a linear operator T is nondegenerate along a direction e. Then T is* A_2*-nondegenerate.*

Indeed, suppose that T is bounded in $\mathrm{L}_2\left(w^{-\frac{1}{2}}\right)$ as in Definition 13. Taking in (11) a family $\mathcal{S} = \{B\}$ consisting of a single ball $B \subset \mathbb{R}^n$,

$X = \mathrm{L}_2\left(w^{-\frac{1}{2}}\right)$ and a nonnegative locally summable function f supported in B yields

$$f_B \left(\int_B w\right)^{\frac{1}{2}} = \|\mathcal{A}_{\mathcal{S}} f\|_X \leqslant c\,\|f\|_X = c\left(\int_B f^2 w\right)^{\frac{1}{2}}. \tag{14}$$

Rearranging the terms of (14), we arrive at

$$(f_B)^2 \leqslant c^2 \frac{1}{\int_B w} \int_B f^2 w,$$

which is a well-known characterization of the Muckenhoupt weights from [27, Chapter 5, §1.4]; setting $f = (w+\varepsilon)^{-1}$ and passing to the limit $\varepsilon \to 0$ quickly leads to (2).

Observe that Proposition 15 also implies that if a suitably nondegenerate operator T acts boundedly in X then all operators $\mathcal{A}_{\mathcal{S}}$ with disjoint collections $\mathcal{S}$ of cubes or balls are uniformly bounded in X. It is not clear in general how this property is related to other properties. One, of course, immediately notices that such operators $\mathcal{A}_{\mathcal{S}}$ are bounded in L_p for both $p = 1$ and $p = \infty$ so their uniform boundedness in a lattice X does not imply per se that X is A_1-regular. However, and somewhat surprisingly, this implication holds true at least in the case of variable exponent Lebesgue spaces if we also assume that X is p-convex and q-concave for some $1 < p, q < \infty$; see [5, Theorem 5.7.2]. This rather involved result together with Proposition 15 provides at once the converse implication $3 \Rightarrow 1$ of Theorem 1 in the case of variable exponent Lebesgue spaces.

Corollary 17. *Suppose that $p(\cdot)$ is a measurable function on $\mathbb{R}^n$ such that $1 < \operatorname{ess\,inf}_{x\in\mathbb{R}^n} p(x) \leqslant \operatorname{ess\,sup}_{x\in\mathbb{R}^n} p(x) < \infty$ and a linear operator T is nondegenerate along a direction and bounded in $\mathrm{L}_{p(\cdot)}$. Then both $\mathrm{L}_{p(\cdot)}$ and $\mathrm{L}_{p'(\cdot)}$ are A_1-regular.*

Now we give a standard condition sufficient for a Calderon-Zygmund operators to be nondegenerate along a direction.

Proposition 18 ([27, Chapter 5, §4.6])**.** *Suppose that T is a Calderon-Zygmund operator with kernel K and there exist some $u \in \mathbb{R}^n$ and a constant c such that for any $x \in \mathbb{R}^n$ and $t \neq 0$ we have*

$$|K(x, x+tu)| \geqslant ct^{-n}. \tag{15}$$

Then T is nondegenerate along the direction $x_0 = su$ with some $s > 0$ and hence T is A_2-nondegenerate.

The two typical examples are the Hilbert transform H on $\mathbb{R}$ with kernel $K(x,y) = \frac{c_1}{x-y}$ and Riesz transforms R_j, $1 \leqslant j \leqslant n$ on $\mathbb{R}^n$ with kernel $K(x,y) = \frac{c_n(y_j - x_j)}{|y-x|^{n+1}}$, where $c_n \neq 0$ are some constants. It is evident that these kernels satisfy condition (15) for $u = e_j$, e_j being the j-th coordinate basis vector of $\mathbb{R}^n$.

For completeness, let us prove Proposition 18. Indeed, condition (3) on the kernel K imply that

$$|K(x, x+r[-x_0+v])-K(x, x-rx_0)| \leqslant C_K \frac{|rv|^\gamma}{|rx_0|^{n+\gamma}} \leqslant c' s^{-n-\gamma} r^{-n} \tag{16}$$

for all $v \in \mathbb{R}^n$, $|v| < \frac{1}{2}|x_0| \wedge 1 = \frac{1}{2}s|u| \wedge 1$, and any $r \neq 0$ with some constant c' independend of s. By taking s sufficiently large we may assume that (16) holds true for all $|v| < 1$. Therefore (15) implies that

$$\begin{aligned}
&|Tf(x)| = \\
&\left|K(x, x - rx_0) \int\limits_B f(y) dy + \int\limits_B [K(x, y) - K(x, x - rx_0)] f(y) dy\right| \\
&\geqslant |K(x, x - rx_0)| \int\limits_B f(y) dy - \left|\int\limits_B [K(x, y) - K(x, x - rx_0)] f(y) dy\right| \\
&\geqslant |K(x, x - rx_0)| \int\limits_B f(y) dy - \int\limits_B |K(x, y) - K(x, x - rx_0)| \, |f(y)| dy \\
&\qquad \geqslant \int\limits_B f(y) dy \cdot \left(c(rs)^{-n} - c' s^{-n-\gamma} r^{-n}\right) = (cs^{-n} - c' s^{-n-\gamma}) f_B
\end{aligned}$$

for any f supported on the ball $B \subset \mathbb{R}^n$ having radius r and centered at the origin and for any $x \in B \pm rx_0$. Choosing s sufficiently large yields (10), so T is indeed nondegenerate along the direction x_0 and is therefore A_2-nondegenerate by Corollary 16. The proof of Proposition 18 is complete.

5. A lemma about A_p-regularity

In this section we establish the following auxiliary result that we will need in Section 6 below.

Theorem 19. *Suppose that X is a Banach lattice of measurable functions on $(S \times \Omega, \mu \times \nu)$ such that X satisfies the Fatou property and X is r-convex with some $r > 1$. Suppose also that*

1. *X is A_p-regular with constants (c_1, m_1) for some $1 < p < \infty$,*
2. *X^δ is A_1-regular with constants (c_2, m_2) for some $\delta > 0$.*

Then lattice X is A_1-regular with constants depending only on the constants of the corresponding A_p-regularity of X, A_1-regularity of X^δ and the value of δ.

The assumption that X is r-convex with some $r > 1$ is actually superflous but it allows us to significantly simplify the technical details of the proof; we will indicate how it can be dropped at the end of

the section. This theorem is easily established from the corresponding result about A_p weights with the help of a fixed point theorem.

Lemma 20. *Suppose that a weight w on $(S\times\Omega, \mu\times\nu)$ satisfies $w \in \mathrm{A}_p$ and $w^\delta \in \mathrm{A}_1$ with some $1 < p < \infty$ and $\delta > 0$. Then $w \in \mathrm{A}_1$ with an estimate for the constants depending only on δ, the corresponding constants of the A_p condition for w and the A_1 condition for w^δ.*

Lemma 20 is essentially a particular case $X = \mathrm{L}_\infty(w)$ of Theorem 19. This result is motivated by a very simple observation: by the factorization of A_p weights (see, e. g., [27, Chapter 5, §5.3]) we have $w = \omega_0\omega_1^{1-p}$ with $\omega_j \in \mathrm{A}_1$, and since we also have $w^\delta \in \mathrm{A}_1$, w is bounded away from 0 on every ball, which suggests that the singularities of the denominator factor ω_1 have to be dominated by the singularities of the nominator factor ω_0 in some sense and ω_1 should essentially cancel out in this factorization. To prove Lemma 20, fix $\omega \in \Omega$ such that $w(\cdot,\omega) \in \mathrm{A}_p$ and $w^\delta(\cdot,\omega) \in \mathrm{A}_1$, and let $B(x,r) \subset S$, $x \in S$, $r > 0$, be an arbitrary ball of S. Then sequential application of the A_p condition satisfied by weight w, the Jensen inequality with convex function $t \mapsto t^{-\delta(p-1)}$, $t > 0$, and the A_1 condition satisfied by the weight w^δ yields

$$
\begin{aligned}
(17)\quad & \frac{1}{\nu(B(x,r))}\int_{B(x,r)} w(u,\omega)d\nu(u) \leqslant \\
& c\left[\frac{1}{\nu(B(x,r))}\int_{B(x,r)} [w(u,\omega)]^{-\frac{1}{p-1}}d\nu(u)\right]^{-(p-1)} = \\
& c\left[\frac{1}{\nu(B(x,r))}\int_{B(x,r)} [w(u,\omega)]^{-\frac{1}{p-1}}d\nu(u)\right]^{-\delta(p-1)\cdot\frac{1}{\delta}} \leqslant \\
& c\left[\frac{1}{\nu(B(x,r))}\int_{B(x,r)} [w(u,\omega)]^{\delta}d\nu(u)\right]^{\frac{1}{\delta}} \leqslant c' w(x,\omega)
\end{aligned}
$$

for almost all $x \in S$ with some constants c and c' depending only on the corresponding constants of the A_p condition for w, the A_1 condition for w^δ and the value of δ. Since ω, x and B are arbitrary, estimate (17) implies that $w \in \mathrm{A}_1$ with the necessary estimates of the constants, which concludes the proof of Lemma 20.

In order to reduce Theorem 19 to Lemma 20 we need to show that under the conditions of Theorem 19 an arbitrary function $f \in X$ has a majorant w such that with the appropriate estimates on the constants w is an A_p-majorant of f in X and simultaneously w^δ is an A_1-majorant of $|f|^\delta$ in X^δ. At a first glance it may seem that there is little reason to suspect existence of a common majorant in sets that look vastly different (for example, a majorant w such that $w^\delta \in \mathrm{A}_1$ may not even be locally summable in the first variable, while on the other hand a majorant $w \in \mathrm{A}_p$ may vanish near some points); however,

careful application of the celebrated Ky-Fan–Kakutani fixed point theorem allows us to establish the existence of a common majorant with relative ease in this setting.

Theorem ([7])**.** *Suppose that K is a compact set in a locally convex linear topological space. Let Φ be a mapping from K to the set of nonempty convex compact subsets of K. If the graph $\{(x, y) \in K \times K \mid y \in \Phi(x)\}$ of Φ is closed in $K \times K$ then Φ has a fixed point, i. e. $x \in \Phi(x)$ for some $x \in K$.*

We will also need the following sets of nonnegative a. e. measurable functions w on $(S \times \Omega, \nu \times \mu)$:

$$BA_p\,(C) = \left\{ w \mid \operatorname*{ess\,sup}_{\omega \in \Omega} \|M\|_{L_p\left(w^{-\frac{1}{p}}(\cdot,\omega)\right)} \leqslant C \right\};$$

$$BA_1\,(C) = \left\{ w \mid \operatorname{ess\,sup} \frac{Mw}{w} \leqslant C \right\}.$$

These are the sets of Muckenhoupt weights with given bounds on the constants ("the Ball of A_p"). Since by our conventions $0 \in BA_p\,(C)$, these sets are nonempty for all $C \geqslant 0$.

Proposition 21 ([25, Proposition 3.4]; see also [10, Lemma 4.2])**.** *Suppose that $1 \leqslant p < \infty$ a. e. and $C \geqslant 0$. The set $BA_p\,(C)$ is a nonempty convex cone which is also logarithmically convex and closed in measure.*

The proof of convexity and closedness is quite routine; the logarithmic convexity is a bit harder but we will not need it under the assumptions of Theorem 19.

We are now ready to prove Theorem 19. Let B be the closed unit ball of X that we need to equip B with a suitable topology. We may assume that $r < \infty$. By Proposition 2 there exists some function $a \in (X^r)'$ such that $\|a\|_{(X^r)'} = 1$ and $a > 0$ almost everywhere. This implies that for any $u \in B$ we have $\int |u|^r a \leqslant \|u^r\|_{X^r} \|a\|_{(X^r)'} = \|u\|_X^r \|a\|_{(X^r)'} \leqslant 1$, so $B \subset L_r\left(a^{-\frac{1}{r}}\right)$ is a bounded subset of $Y = L_r\left(a^{-\frac{1}{r}}\right)$. Since X satisfies the Fatou property, B is closed in measure, and therefore B is a closed bouned set in Y. Equipping Y with the weak topology makes B a compact set.

Observe that since A_1-regularity of X implies A_1-regularity of X^β for all $0 < \beta < 1$ we may assume that $0 < \delta < 1$, otherwise the conclusion of Theorem 19 is immediate. It is easy to verify in the same way as in Proposition 21 that the set $[BA_1\,(C)]^{\frac{1}{\delta}} = \{w \mid w^\delta \in BA_1\,(C)\}$ is convex using the fact that $(a + b)^\delta \leqslant a^\delta + b^\delta$ for all numbers $a, b \geqslant 0$, and closedness in measure of $[BA_1\,(C)]^{\frac{1}{\delta}}$ follows immediately from Proposition 21.

We define a map Φ on $B \times B$ acting into subsets of $B \times B$ by

$$\Phi((u,v)) = \{u_1, v_1 \in X \mid \\ u_1 \in B \cap B\mathrm{A}_p\,(c_1)\,, v_1 \in B \cap [B\mathrm{A}_1\,(c_2)]^{\frac{1}{\delta}}\,, \\ f \vee \frac{1}{2}(u \vee v) \leqslant A\,(u_1 \wedge v_1)\Big\}$$

with $A = m_1 \vee m_2^{\frac{1}{\delta}}$. Nonemptiness of the values of Φ is implied by the A_p-regularity of X and A_1-regularity of X^δ in the conditions of Theorem 19. The condition $f \vee \frac{1}{2}(u \vee v) \leqslant A\,(u_1 \wedge v_1)$ is of course equivalent to (and a shorthand for) the six inequalities $f \leqslant Au_1$, $f \leqslant Av_1$, $u \leqslant 2Au_1$, $v \leqslant 2Au_1$, $u \leqslant 2Av_1$ and $v \leqslant 2Av_1$. It is easy to see using Proposition 21 and remarks after it that the graph of Φ is a convex set that is closed with respect to the convergence in measure. Let us verify that the graph Γ of Φ is closed in $Y \times Y$. Indeed, the weak topology of Y is metrizable on a bounded set B. If $x_j \in \Gamma$ and $x_j \to x \in Y$ then there exists some sequence y_j of convex combinations of x_j such that $y_j \to x$ in the strong topology of Y, and $y_j \in \Gamma$ by the convexity of Γ. Strong convergence in Y implies convergence in measure, so $y_j \to x$ in measure. Since Γ is closed in measure, it follows that $x \in \Gamma$ and thus Γ is indeed closed in $Y \times Y$. We also infer that the values of Φ are convex and closed in the compact set $B \times B$ and thus they are compact in $Y \times Y$.

By the Ky Fan–Kakutani fixed point theorem there exists some $(u,v) \in B \times B$ such that $(u,v) \in \Phi((u,v))$. This implies that u and v are pointwise equivalent to one another and so $w = Au$ is a majorant of f such that $w \in \mathrm{A}_p$ and $w^\delta \in \mathrm{A}_1$ with the appropriate estimates of the constants. By Lemma 20 it follows that $w \in \mathrm{A}_1$ with suitable estimates on the constants, which concludes the proof of Theorem 19.

We now briefly explain how the assumption that X is r-convex can be dropped from Theorem 19. By using [25, Proposition 3.6] it is sufficient to look for a majorant on an increasing sequence of sets $A_j \subset S \times \Omega$ such that $\bigcup_j A_j = S \times \Omega$. Let A_j be such a sequence of sets of finite measure, and let $B_j = A_j \cap \left\{\frac{1}{j} \leqslant f \leqslant j, \frac{1}{j} \leqslant a \leqslant j\right\}$; then B_j is also an increasing sequence of sets of finite measure such that $\bigcup_j B_j = S \times \Omega$ up to a set of measure 0. Now we only have the inclusion $X \subset \mathrm{L}_1\,(a)$ for some $a \in X'$, $a > 0$ a. e., so the set B defined as before may not necessarily be compact. However, simple estimates show that

$$D = \{\chi_{B_j} \log g \mid g \in B,\, g \geqslant f\}$$

is a bounded set in L_2, and since it is closed in measure, it is also compact in the weak topology of L_2. Passing to the logarithms, we define the map Φ on D instead of B, and the rest of the proof works

as before except that we need to use the logarithmic convexity of sets $B\mathrm{A}_p(C)$ and of the unit ball of X when we invoke Proposition 21.

6. Necessity of A_1-regularity

In this section we establish the converse implication $3 \Rightarrow 1$ of Theorem 1. We will need the following fairly well known result, the proof of which in the present setting can be found in [25, Theorem 2.6].

Theorem 22. *Suppose that Y is a Banach lattice on $(S\times\Omega, \nu\times\mu)$ with an order continuous norm. If a linear operator T is bounded in $Y^{\frac{1}{2}}$ then for every $f \in Y'$, $m > 1$ and $a > K_G \|T\|_{Y^{\frac{1}{2}}}$, K_G being the Grothendieck constant, there exists a majorant $w \geqslant |f|$, $\|w\|_{Y'} \leqslant \frac{m}{m-1}\|f\|_{Y'}$, such that $\|T\|_{\mathrm{L}_2\left(w^{-\frac{1}{2}}\right)} \leqslant a\sqrt{m}$.*

This theorem essentially says that for suitably nondegenerate operators T boundedness of T in a lattice $Y^{\frac{1}{2}}$ implies that Y' is A_2-regular, which binds the boundedness property of certain operators in a lattice back to a regularity property for some related lattices. The proof of Theorem 22 given in [25, §6] is merely a slight refinement of the proof of [12, Theorem 3.5], which is in turn a variant of the well-known Maurey–Krivine factorization theorem (see [21]). For the first time these ideas were exploited in a similar context in [24].

Theorem 23. *Suppose that X is a Banach lattice of measurable functions on $(S\times\Omega, \nu\times\mu)$ such that X is p-convex and q-concave for some $1 < p, q < \infty$ and X satisfies the Fatou property. Let T be a linear operator on $\mathrm{L}_2(S\times\Omega)$ such that both T and T^* are A_2-nondegenerate and T acts boundedly in X and in all L_s for $1 < s < \infty$. Then lattices X and X' are A_1-regular.*

Let us now prove Theorem 23. By the p-convexity condition X^p is also a Banach lattice with the Fatou property, and so $X^{p(1-\theta)}\mathrm{L}_t^\theta$ is also a Banach lattice for all $1 \leqslant t \leqslant \infty$ and $0 < \theta < 1$. Choosing $\theta = 1 - \frac{1}{p}$ shows that $Y_s = X\mathrm{L}_s$ is a Banach lattice for all sufficiently large s. Lattice Y_s satisfies the Fatou property and has order continuous norm (because L_s has order continuous norm for $s < \infty$). Since T is bounded in X and in L_s for all $1 < s < \infty$, by the interpolation theorem mentioned in Section 1 operator T is also bounded in $X^{\frac{1}{2}}\mathrm{L}_s^{\frac{1}{2}} = Y_s^{\frac{1}{2}}$ for all $1 < s < \infty$. Theorem 22 and A_2-nondegeneracy of T then imply that lattice $Y_s' = X'\mathrm{L}_{s'}$ is A_2-regular for all sufficiently large s. By Theorem 6 it follows that lattice X' is A_3-regular, and furthermore by Proposition 5 lattice $X^{\frac{1}{3}}$ is A_1-regular. Since the convexity assumptions of Theorem 23 imply that lattices X and X' have order continuous norm, we have $X' = X^*$ and $X = (X')^*$, and moreover $X \cap \mathrm{L}_2$ is dense in X and $X' \cap \mathrm{L}_2$ is dense in X', so the duality relation $\int (Tf)g = \int f(T^*g)$ for $f \in X \cap \mathrm{L}_2$ and $g \in X' \cap \mathrm{L}_2$ shows that boundedness of T in X

implies boundedness of the conjugate operator T^* in X' and vice versa. Repeating the argument above with lattice X' in place of X (which is a q'-convex lattice since X is q-concave) and operator T^* in place of T shows that lattice X is A_3-regular and lattice $(X')^{\frac{1}{3}}$ is A_1-regular. Finally, applying Theorem 19 to X and to X' with $p = 3$ and $\delta = \frac{1}{3}$ shows that lattices X and X' are both A_1-regular, which concludes the proof of Theorem 23.

Theorem 23 implies the converse implication $3 \Rightarrow 1$ of Theorem 1 because the Riesz projection R_j is an A_2-nondegenerate self-adjoint Calderon-Zygmund operator which is nondegenerate by Proposition 18, and so R_j satisfies the conditions of Theorem 23.

We remark that the assumptions of p-convexity and q-concavity could be dropped from Theorem 23 if we assume boundedness of T in $X^{\frac{1}{2}}$ and $(X')^{\frac{1}{2}}$ instead of just X and get rid of p-convexity assumption in Theorem 19 as described at the end of Section 5. It is, however, unclear whether boundedness of a Calderon-Zygmund operator T in X implies its boundedness in $X^{\frac{1}{2}}$; it seems plausible because T acts boundedly from L_∞ to BMO, but as far as I know, all available interpolation results that make it possible to replace L_∞ by BMO as an endpoint in the appropriate interpolation scale (see, e. g., [13], [26]), which are, however, true for all linear operators and not just Calderon-Zygmund, work only under the assumption that both X^α and $(X')^\alpha$ are A_1-regular for some $\alpha > 0$, which is part of what we are trying to establish in this setting.

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